# THE ELLIS-JAFFE SUM RULE: THE ESTIMATES OF THE NEXT-TO-NEXT-TO-LEADING ORDER QCD CORRECTIONS

**A.L. Kataev**$^{*)}$

Theoretical Physics Division, CERN
CH - 1211 Geneva 23

## ABSTRACT

The procedure of the estimates of the higher-order perturbative QCD corrections to the physical quantities is generalized to the case when the quantities under consideration obey the renormalization group equations with the corresponding anomalous dimension functions. This procedure is used to estimate the $\alpha_s^3$-corrections to the singlet part of the Ellis-Jaffe sum rule for $f = 3$ numbers of flavours.

---

$^{*)}$ On leave of absence from Institute for Nuclear Research, Moscow 117312, Russia.

CERN-TH.7333/94
July 1994

# 1

The best way of controlling the theoretical uncertainties of the perturbative QCD predictions is the direct analytical or numerical calculation of the concrete terms in the corresponding perturbative series. However, after the results of the calculations of the next-to-next-to-leading order (NNLO) corrections to the number of physical quantities became available [1]–[2], experimentalists and theoreticians started to be interested in the effects of still uncalculated higher-order terms. In the work of Ref. [6], two "optimal" methods of fixing the renormalization scheme ambiguities were used to estimate the next-after-next-to-next-to-leading order (NANNLO) corrections to certain renormalization-group invariant quantities. These methods were the principle of minimal sensitivity (PMS) [6] and the effective charges approach (ECH) [7], which is known to be a posteriori identical to the so-called scheme-invariant perturbation theory [8]. The quantities studied in Ref. [5] are the $e^+e^-$-annihilation ratio $R(s)$, the $\tau$-lepton decay ratio $R_E$, and the Bjorken non-polarized and polarized sum rules.

However, the quantities obeying the renormalization group equations with anomalous dimension functions were not considered in Ref. [5]. In this note we will fill in this gap and apply the generalization of the ideas used in Ref, [5] to estimate the higher-order corrections to the singlet part of the Ellis-Jaffe sum rule (EJSR), recently calculated at the $0(\alpha_s^2)$ order [10]. We present the concrete $0(\alpha_s^2)$estimates for $f = 3$ numbers of flavours in two related forms, namely in the factorization-scheme-invariant form and in the form that necessitates the application of an additional guess about the value of the unknown four-loop coefficient of the corresponding singlet anomalous dimension function.

# 2

The experimental measurements [10] of the structure function $g_1^{P(n)}$ of the polarized deep-inelastic lepton-nucleon scattering has stimulated number of works aimed at a theoretical study of the structure function $g_1^{P(n)}$ and its first moment, namely the EJSR (see e.g., [11]–[15]).

The theoretical expression for the EJSR can be presented in the following form:

$$EJSR(Q^2) = \int_0^1 g_1^{P(n)}(x, Q^2)dx = EJ_{NS} + EJ_{SI}(Q^2) \; . \tag{1}$$

The non-singlet contribution to this sum rule is defined as

$$EJ_{NS}(Q^2) = (1 - a - \sum_{i\geq 2} d6NS_i a^i)\;(\pm\frac{1}{12}a_3 + \frac{1}{36}a_8)\;, \tag{2}$$

where $a - \alpha s/\pi$, $a_3 = \Delta u - \Delta d$, $a_8 = \Delta u + \Delta d - 2\Delta S$ and $\Delta u, \Delta d, \Delta s$ can be interpreted as the measure of the polarization of quarks in a nucleon.

The singlet contribution to Eq. (1) reads

$$EJ_{SI}(Q^2) = C(a)\exp\{-\int^{a(Q^2)} \frac{\gamma_{SI}(x)}{\beta(x)} dxy \frac{1}{9} \Delta \sum_{inv}^{\alpha}\;, \tag{3}$$

where

$$\delta\sum_{inv}^{\alpha}=\exp\{-\sum^{a(\mu^2)}\frac{\gamma_{SI}(x)}{\beta(x)}dxy\Delta\sum(\mu^2)$$

and $\Delta\Sigma=\Delta u+\Delta d+\Delta s$. The first three coefficients of the QCD $\beta$-function

$$\beta(a)=-\sum_{i\geq 0}\beta_i a^{i+2} \tag{4}$$

and of the anomalous dimension of the singlet axial current

$$\gamma_{SI}(a)=\sum_{i\geq 0}\gamma_i a^{i+1} \tag{5}$$

are known in the $\overline{MS}$ scheme from the results of calculations [16], [17]. They have the following numerical form:

$$\begin{aligned}\beta_0 &= 2.75-0.167\ f\\ \beta_1 &= 6.375-0.792\ f\\ \beta_2 &= 22.320-4.369\ f+0.094\ f^2\end{aligned} \tag{6}$$

and

$$\begin{aligned}\gamma_0 &\equiv 0\\ \gamma_1 &= -0.5\ f\\ \gamma_2 &= -2.458\ f+0.028\ f^2\ .\end{aligned} \tag{7}$$

The coefficient $\gamma_3$ is unknown and we will need to somehow fix its value in the process of further considerations.

The perturbative expression for the coefficient function

$$C(a)=1+\sum_{i\geq 1}r_i a^i \tag{8}$$

is explicitly known in the $\overline{MS}$ scheme at the $o(a^2)$ level [8], [9]. The results of direct calculations are

$$\begin{aligned}r_1 &= -1\\ r_2 &= -4.583+1.162\ f\ .\end{aligned} \tag{9}$$

Our aim is to estimate the value of the coefficient $r_3$, in the $\overline{MS}$, thus roughly fixing the uncertainties due to the lack of knowledge of the explicitly unknown correction to the singlet part of the EJSR at the $0(\alpha_s^3)$ level, already achieved in the direct calculations of the related non-singlet contribution [4]. This estimate will be obtained with the help of a procedure that is analogous to the one used in Ref. [5] to estimate the $0(\alpha_s^4)$ contribution to the non-singlet part of the EJSR.

Let us follow the considerations of Ref. [19] and define the renormalization-group-invariant quantity:

$$\begin{aligned} R_{EJ_{SI}} &= +Q^2\,\frac{d\ln[E]_{SI}(Q^2)}{dQ^2} \\ &= \gamma(a)+\beta(a)\,\frac{\partial C(a)/\partial a}{C(a)} \\ &= d_0^{SI}a^2(1+d_1^{SI}a+d_2^{SI}a^2) \end{aligned} \tag{10}$$

where

$$\begin{aligned} d_0^{SI} &= -\beta_0 r_1+\gamma_1 \\ d_0^{SI}d_1^{SI} &= -2\beta_0 r_2+\beta_0 r_1^2\beta_1 r_1+\gamma_2 \\ d_0^{SI}d_2^{SI} &= -3\beta_0 r_3+\gamma_3 = f(\beta_0,\beta_1,\beta_2,r_1,r_2) \end{aligned} \tag{11}$$

and

$$f(\beta_0,\beta_1,\beta_2,r_1,r_2)=\beta_0 r_1^3-3\beta_0 r_1 r_2+2\beta_1 r_2-\beta_1 r_1^2+\beta_2 r_1\ . \tag{12}$$

Using the ideas of Refs. [20], [5], one can insert the values of the coefficients $d_0$, $d_1$ and $c_1=\beta_1/\beta_0$ into the following expression:

$$d_0^{SI}d_2^{SI}=d_0^{SI}d_1^{SI}(\frac{3}{4}d_1^{SI}+C_1)\ . \tag{13}$$

This can be obtained as the residual term in the re-expansion of the effective charge $R_{EJ_{SI}}\equiv d_0^{SI}a_{ECM}^2$ in terms of the coupling constant of the initial scheme, in our case the $\overline{MS}$ scheme [1]. We consider this term as the one that simulates the contribution of the non-calculated NNLO correction.

As the result of application of this procedure we get the following expression of the coefficient $r_3$:

$$r_3=-\frac{d_0^{SI}d_2^{SI}}{3\beta_0}+\frac{\gamma_3}{3\beta_0}-\frac{f(\beta_0,\beta_1,\beta_2,r_1,r_2)}{3\beta_0}\ , \tag{14}$$

where the anomalous-dimension term $\gamma_3/3\beta_0$ remains unknown. Note that this unknown term cancels in the expression for Eq. (3) after taking into account the corresponding expansion of the anomalous-dimension term

$$\begin{aligned} &\exp\left\{-\int^a\frac{\gamma_{SI}(x)}{\beta(x)}dx\right\}=1-\frac{\gamma_1}{\beta_0}a \\ &+\left[\frac{\gamma_1^2}{\beta_0^2}-\frac{\gamma_2}{\beta_0}+\frac{\gamma_1\gamma_2}{\beta_0^2}\right]\frac{1}{2}a^2 \\ &+\left[-\frac{\gamma_1^3}{\beta_0^3}+\left(\frac{\gamma_1\gamma_2}{\beta_0^2}-\frac{\gamma_1^2\beta_1}{\beta_0^3}\right)\frac{3}{2}-\left(\frac{\gamma_3}{\beta_0}-\frac{\gamma_1\beta_2}{\beta_0^2}+\frac{\gamma_1\beta_1^2}{\beta_0^3}-\frac{\gamma_2\beta_1}{\beta_0^2}\right)\right]\frac{a^3}{3}\ . \end{aligned} \tag{15}$$

[1]The result of re-expansion of a similar expression obtained within the framework of the PMS approach will differ from Eq. (13) only slightly.

Taking now $f = 3$ numbers of flavours and using Eqs. (5), (6), (8), (10), (12) and (14), we obtain the following expression for the singlet contribution to the EJSR:

$$EJ_{SI}(Q^2) = \left[1 - 0.33a = 0.55a^2 - \underline{\underline{2}}a^3\right]\frac{1}{9}\ \Delta\sum_{inv}^{\alpha} , \tag{16}$$

where the $0(a)$ and $0(a^2)$ corrections are known from the results of explicit calculations (see E.g., Ref. [9]) and the $0(a^3)$ contribution is our estimate of the value of a correction that is still not calculated. Notice that the possibility to apply the above-described approach for the estimates of the $0(\alpha_s^3)$ corrections to the singlet part of the EJSR is supported by the good agreement of the results of similar estimatges of the $0(\alpha_s^3)$ corrections to the corresponding non-singlet contribution for $f = 3$ numbers of flavours [5] with the results of the explicit calculations of Ref. [4].

In order to get a similar estimate for the $Q^2$-dependent normalization of $\Delta\Sigma$, namely for the case when $\Delta\Sigma = \Delta\Sigma(\mu^2 = Q^2)$, it is necessary to somehow fix the value of the unknown $\gamma_3$ term of the singlet anomalous dimension. For $f = 3$, we will use the following bold guess-estimate:

$$\gamma_3 \approx \frac{\gamma_2^2}{\gamma_1} \approx -34 , \tag{17}$$

with which we obtain the expression for the singlet contribution to the EJSR in the case when the $Q^2$ dependence of $\Delta\Sigma$ is specified:

$$EJ_{SI}(Q^2) = \left[1 - a - 1.09a^2 \leq 4a^3\right]\ \frac{1}{9}\ \Delta\Sigma(Q^2) . \tag{18}$$

Note, however, that we are unable to present a similar estimate of the $0(\alpha_s^3)$ correction to the singlet part of the EJSR for $f = 4$ numbers of flavours. Indeed, for the case of $f = 4$ the value of the coefficient $d_0^{SI}$ in Eqs. (10) and (11) is almost nullified ($d_0^{SI} \approx 0$ since $r_1 \approx \gamma_1/\beta_0$). Therefore, it is impossible to determine the value of the corresonding coefficient $d_0^{SI}, d_1^{SI}$, presented in Eq. (11). This example demonstrates the limitations of the procedure discussed above.

# 4

To conclude, we touched the problem of fixing the uncertainties due to still-uncalculated $0(\alpha_s^3)$ corrections to the singlet contribution into the EJSR, for $f = 3$. Combining the obtained estimates with the results of available NNLO calculation [4] of the non-singlet contributions into the EJSR and with the corresponding NANNLO estimates [5], we arrive at the following expressions for the EJSR, related to $f = 3$:

$$\begin{aligned}\int_0^1 g_1^{p(n)}(x,Q^2)dx \;&=\; \left[1 - a - 3.58a^2 - 20.21a^3 - 130a^4\right] \times \left(\pm\frac{1}{12}a_3 + \frac{1}{36}a_8\right) \\ &+\left[1 - 0.33a = 0.55a^2 - 2a^3\right]\ \frac{1}{9}\ \Delta\sum_{inv}^{\alpha} ,\end{aligned} \tag{19}$$

or

$$\int_0^1 g_1^{p(n)}(x,Q^2)dx = \left[1-a-3.58a^2-20.21a^3-130a^4\right]\times\left(\pm\frac{1}{12}a_3+\frac{1}{36}a_8\right) + \left[1-a-1.09a^2-4a^3\right]\frac{1}{9}\,\Delta\sum(Q^2)\ . \tag{20}$$

It can be seen that the perturbative contributions to the singlet part of the EJSR, including the $0(\alpha_s^3)$ term that we estimated, are negative. They are significantly smaller than the coefficients of the perturbative series of the non-singlet part, which include the results of the concrete $0(\alpha_s^3)$ calculations [4] and the estimates of the $0(\alpha_s^4)$ terms [5].

**Acknowledgements**

We are grateful to J. Ellis and M. Karliner for the interest in the work done in collaboration with V.V. Starshenko [5] and for useful discussions, which stimulated the considerations presented in this note.

# References

[1] S.G. Gorishny, A.L. Kataev and S.A. Larin, in "Standard Model and Beyond", Proc. Int. Workshop JINR-CERN-IHEP, Dubna, 1-5 October 1990, eds. S. Dubnicka, D. Ebert and A. Sazonov (World Scientific, Singapore, 1991), p. 288; *Phys.Lett.* **B259** (1991) 144.

[2] L.R. Surguladze and M.A. Samuel, in "Beyond the Standard Model II", Norman, OK, 1-3 November 1990, eds. K. Mioton, R. Kantowski and M.A. Samuel (World Scientific, Singapore, 1991), p. —; *Phys.Rev.Lett.* **66** (1991) 560; **66** (1991) 2416 (Erratum).

[3] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren *Phys.Rev.Lett.* **66** (1991) 862.

[4] S.A. Larin and J.A.M. Vermaseren, *Phys.Lett.* **B259** (1991) 345.

[5] A.L. Kataev and V.V. Starshenko, CERN Preprint TH. 7190/94 (1994), hp-ph/9405294.

[6] P.M. Stevenson, *Phys.Rev.* **D23** (1981) 2916.

[7] G. Grunberg, *Phys.Lett.* **B221** (198) 70; *Phys.Rev.* **D29** (1984) 2315.

[8] A. Dhar and V. Gupta, *Phys.Rev.* **D29** (1984) 2822;
V. Gupta, D.V. Shirkov and O.V. Tarasov, *Int.J.Mod.Phys.* **A6** (1991) 3381.

[9] S.A. Larin, CERN Preprint TH. 7208/94 (1994), hep-ph/9403383.

[10] EMC Collaboration, J. Ashman et al., *Nucl.Phys.* **B328** (1989) 1;
SMC Collaboration, B. Adeva et al., *Phys.Lett.* **B302** (1993) 533;

E142 Collaboration, P.L. Anthony et al., *Phys.Rev.Lett.* **71** (1993) 959;
SMC Collaboration, D. Adams et al., *Phys.Lett.* **B329** (1994) 399;
E143 Collaboration, R. Arnold et al., Preliminary results presented by L. Stuart at the Symposium "Radiative Corrections: States and Outlook", Galtinburg, TN, June 1994.

[11] A.V. Efremov and O.V. Teryaev, Dubna report JINR E2-88-287 (1988);
G. Altarelli and G. Ross, *Phys.Lett.* **B212** (1988) 391.

[12] J. Ellis and M. Karliner, *Phys.Lett.* **B213** (1988) 131.

[13] G. Altarelli, P. Nason and G. Ridolfi, *Phys.Lett.* **B320** (1994) 152; **B325** (1994) 538 (Erratum).

[14] S. Narison, G.M. Shore and G. Veneziano, Preprint PM/94-14, SWAT-94/12, CERN Preprint TH. 72222223/94 (1994).

[15] J. Ellis and M. Karliner, CERN Preprint TH. 7324/94 (1994), TAUP-2178-94 (1994), hep-ph/—- and references therein.

[16] O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *Phys.Lett.* **(3B** (1980) 429;
S.A. Larin and J.A.M. Vermaseren, *Phys.Lett.* **B303** (1993) 334.

[17] S.A. Larin, *Phys.Lett.* **B303** (1993) 113;
K.G. Chetyrkin and J.H. Kühn, *Z.Phys.* **C60** (1993) 497.

[18] E.B. Zijlstra and W.L. van Neerven, *Nucl.Phys.* **B417** (1994) 61.

[19] A.L. Kataev, in Proc. QCD-90 Conference, Montpellier, 1990; ed. S. Narison; *Nucl.Phys.* B, Proc.SUppl. **23B** (1991) 72.

[20] J. Kubo and S. Sakakibara, *Z.Phys.* **C14** (1982) 345;
V.V. Starshenko and R.N. Faustov, *JINR Rapid Communications* **7** (1985) 39.